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Citation: American Journal of Physics **90**, 908 (2022); doi: 10.1119/5.0062969
View online: https://doi.org/10.1119/5.0062969
View Table of Contents: https://aapt.scitation.org/toc/ajp/90/12
Published by the American Association of Physics Teachers

A simple electronic circuit demonstrating Hopf bifurcation for an advanced undergraduate laboratory

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(Received 10 July 2021; accepted 26 September 2022)

A nonlinear electronic circuit comprising of three nodes with a feedback loop is analyzed. The system has two stable states, a uniform state and a sinusoidal oscillating state, and it transitions from one to another by means of a Hopf bifurcation. The stability of this system is analyzed with nonlinear equations derived from a repressilator-like transistor circuit. The apparatus is simple and inexpensive, and the experiment demonstrates aspects of nonlinear dynamical systems in an advanced undergraduate laboratory setting.
https://doi.org/10.1119/5.0062969

I. INTRODUCTION

A simple electronic circuit with a nonlinear electronic component can exhibit chaotic behavior.[1–3] Mishina *et al.* used a simple analog electronic circuit and demonstrated successive bifurcations leading to chaotic regimes.[4] Later Hellen used a slightly improved electronic circuit based on a simple nonlinear system, a finite difference equation with a quadratic return map, and showed bifurcation diagrams on an oscilloscope.[5] Goswami and Ray used a simple electronic circuit with a voltage-controlled current source, which exhibits feature-rich dynamics and several bifurcations.[6] Sack *et al.* constructed an electronic circuit which possessed remarkable organization in periodic oscillations in parameter space.[7] Cabeza *et al.* developed an electronic circuit with broad cycles.[8] Even an electronic circuit with a single transistor can display nonlinear behavior like limit cycles such as in the Hartley and Colpitts oscillators.[9]

The Hopf bifurcation is a special kind of bifurcation that involves the transition from stable behaviour of a system to periodic and vice versa.[1] It is prevalent in many physical phenomena, including repressilators in genetic networks, inverted pendulums, fluid flow about a sphere, and electronic circuits with negative differential resistance components.[10–14] Particularly, it is from the repressilator-based configuration that we get our motivation for the circuit. A repressilator is a genetic regulatory network comprised of at least one feedback loop of at least three genes. Taking inspiration from this, Rim *et al.* constructed an electronic circuit with a negative feedback loop consisting of three nodes, where each node models a gene.[15] The primary component of each node is a transistor, which serves as the means for negative feedback between nodes. We present a detailed analysis of the dynamics of the Hopf bifurcation in this 3-node electronic circuit. We obtain explicit formulae for the dependence of the bifurcation resistance on the applied voltage and for the variation of the fixed point voltage with resistance and applied voltage. The transistors used here are inexpensive, and the other components are easily found in any undergraduate physics laboratory. The theory of the Hopf bifurcation is also relatively simple, requiring only knowledge of multivariable calculus and calculus-based classical mechanics. Coupled with the simplicity of the circuit design and the circuit analysis, this experiment is appropriate to introduce advanced undergraduate students to the fascinating world of nonlinear dynamics.

II. STRUCTURE OF THE CIRCUIT

The electronic circuit we use here has been taken from the paper by Rim *et al.*[15] It consists of three identical nodes, which are connected to each other in a cycle to form a negative feedback loop. The structure of each node is shown in Fig. 1, where the transistor, resistors R_1 and R_2, and the capacitor all have fixed values. The resistor R_3 is variable and is the independent bifurcation parameter of the system. The transistor used in a node is an NPN bipolar junction transistor (BJT) with V_B, V_C, and V_E as base, collector, and emitter voltages, respectively, and I_B, I_C, and I_E as corresponding currents. V_i, I_i, V_o, and I_o represent input and output voltages and currents, respectively, and V_{CC} is the supply voltage. In the analysis, all the time-dependent parameters are written in corresponding lower case letters. The entire circuit comprising all the three nodes is shown in Fig. 2. So, the cycle is formed by:

- The v_o of the first node serving as the v_i of the second node;
- The v_o of the second node serving as the v_i of the third node;
- The v_o of the third node serving as the v_i of the first node.

In the whole loop, each of the three resistances labelled R_3 have the same values at all times and are varied simultaneously.

III. CIRCUIT ANALYSIS

A. Modelling of an NPN BJT in active mode

An NPN BJT has four modes of operation: active, cutoff, reverse active, and saturation, which depend on whether $v_{BE} = v_B - v_E$ and $v_{CB} = v_C - v_B$ are positive or negative, where v_{BE} and v_{CB} are the transient voltages at the base-emitter and collector-base junctions, respectively. In our experimental setup, BJT is in the active mode (that is, both v_{BE} and v_{CB} are positive). So, we focus on this mode only here. For an NPN BJT in the active mode, the Ebers-Moll model gives, under the assumption of physically realistic voltages, the equations

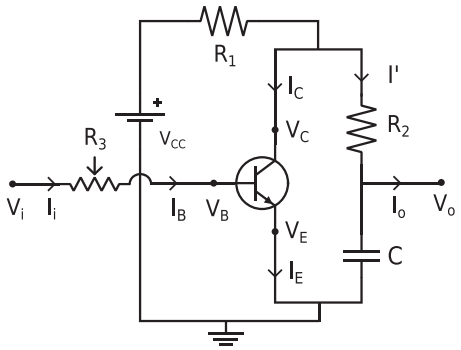

Fig. 1. Circuit diagram of one node showing the components and parameters. R_3 is the variable input resistor, which is also used as the feedback resistor when different nodes are connected together.

$$i_C = I_S \exp \frac{v_{BE}}{V_T}, \tag{1}$$

$$i_C = \alpha i_E, \tag{2}$$

$$i_C = \beta i_B. \tag{3}$$

Here, I_S is the saturation current, V_T is the thermal voltage, and α and β are two proportionality constants. Now, we also have $i_E = i_C + i_B$, with all three positive (as v_{BE} and v_{CB} are positive). Thus, we get the relations $\alpha = \beta/(1+\beta)$ or $\beta = \alpha/(1-\alpha)$ between α and β. For typical NPN BJTs, β ranges from 50 to 200, and so α ranges from 0.98 to 0.99. Also, I_S is of the order of 10^{-12}A for typical NPN BJTs, and $V_T = 0.025875$ V at room temperature (298 K). For further details on NPN BJTs, readers are encouraged to refer to Ref. 9.

B. Lambert *W* function

The Lambert W function $W(x)$ is defined as the real solution of the equation $ye^y = x$. We summarize here some properties of this function for positive x, which will be used throughout. For detailed proofs, one may look at Ref. 17.

- **Single-valued:** The function $W(x)$ is single valued, that is, for any x, there is only one real y such that $ye^y = x$.
- **Positive:** For $x > 0$, $W(x) > 0$.
- **Monotonic:** $W(x)$ is a monotonically increasing function. That is $x_1 > x_2 \Rightarrow W(x_1) > W(x_2)$.
- **Derivative:**

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{1}{x}\frac{W(x)}{1+W(x)}. \tag{4}$$

C. Modelling the node

For this circuit, as the emitter is grounded in the node, $v_E = 0$, and so $v_B = v_{BE}$. Applying Kirchoff's Laws and the results in Sec. III A, we get

$$\frac{\mathrm{d}v_o}{\mathrm{d}t} = \frac{V_{CC} - v_o - i_C R_1 - i_o(R_1+R_2)}{C(R_1+R_2)}, \tag{5}$$

$$\log \frac{i_C}{I_S} = \frac{v_i}{V_T} - \frac{R_3}{\beta V_T} i_C. \tag{6}$$

Let us take $a = I_S R3/\beta V_T$ for simplicity. Then, we can solve for i_C with the help of the Lambert W function $W(x)$,

$$i_C = \frac{\beta V_T}{R_3} W\left(a \exp \frac{v_i}{V_T}\right). \tag{7}$$

Note that a depends only on R_3 (the bifurcation parameter). On substituting this in Eq. (5), we get the variation of v_o as

$$\frac{\mathrm{d}v_o}{\mathrm{d}t} = \frac{V_{CC} - v_o - \beta V_T \frac{R_1}{R_3} W\left(a \exp \frac{v_i}{V_T}\right) - i_o(R_1+R_2)}{C(R_1+R_2)}. \tag{8}$$

D. Modelling the circuit

For this circuit, the input of a node is the output of the previous node. So, the values of V and I of a node depend on both the preceding and succeeding nodes. Now, formally, let us define node 4 to be identical to node 1, and node 0 as identical to node 3. Then, we can say that node n depends on the nodes $n-1$ and $n+1$. Thus, we get for current

$$i_{C_n} = \frac{\beta V_T}{R_3} W\left(a \exp \frac{v_{n-1}}{V_T}\right), \tag{9}$$

$$i_n = i_{B_{n+1}}, \tag{10}$$

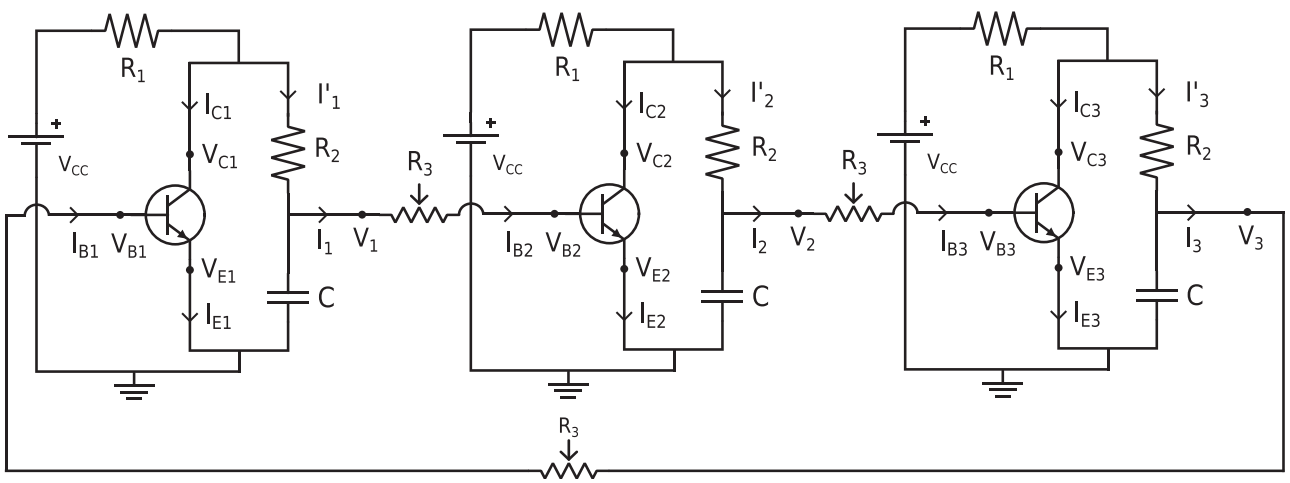

Fig. 2. Circuit diagram of the complete circuit having three nodes and feedback loop.

$$= \frac{V_T}{R_3} W\left(a \exp \frac{v_n}{V_T}\right), \tag{11}$$

and, hence, for voltage

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = \frac{V_{CC} - v_n}{C(R_1 + R_2)} - \frac{V_T}{CR_3} \times \left[W\left(a \exp \frac{v_n}{V_T}\right) + \beta \frac{R_1}{R_1 + R_2} W\left(a \exp \frac{v_{n-1}}{V_T}\right)\right]. \tag{12}$$

IV. MATHEMATICAL ANALYSIS OF THE MODEL

Due to the nonlinearity, we are unable to find a closed form solution for this system. Instead, we analyze the behaviour of the stable states of the system.

A. Fixed point

The system has a fixed point when $\mathrm{d}v_n/\mathrm{d}t = 0$ for $n = 1, 2, 312$. That is

$$V_{CC} = v_1 + V_T \left[\frac{R_1 + R_2}{R_3} W\left(a \exp \frac{v_1}{V_T}\right) + \beta \frac{R_1}{R_3} W\left(a \exp \frac{v_3}{V_T}\right)\right]; \tag{13}$$

$$V_{CC} = v_2 + V_T \left[\frac{R_1 + R_2}{R_3} W\left(a \exp \frac{v_2}{V_T}\right) + \beta \frac{R_1}{R_3} W\left(a \exp \frac{v_1}{V_T}\right)\right]; \tag{14}$$

$$V_{CC} = v_3 + V_T \left[\frac{R_1 + R_2}{R_3} W\left(a \exp \frac{v_3}{V_T}\right) + \beta \frac{R_1}{R_3} W\left(a \exp \frac{v_2}{V_T}\right)\right]. \tag{15}$$

1. Equality of $_n$ at fixed point

Without loss of generality, we can assume that $v_1 \leq v_2 \leq v_3$ at the fixed point. Subtracting Eq. (14) from Eq. (15), we get

$$(v_3 - v_2) + V_T \frac{R_1 + R_2}{R_3} \left[W\left(a \exp \frac{v_3}{V_T}\right) - W\left(a \exp \frac{v_3}{V_T}\right)\right] = V_T \beta \frac{R_1}{R_3} \left[W\left(a \exp \frac{v_1}{V_T}\right) - W\left(a \exp \frac{v_2}{V_T}\right)\right]. \tag{16}$$

The function $W(x)$ is monotonic for positive arguments. So, as $v_3 \geq v_2$ by assumption, the left hand side of the equation is ≥ 0. Yet, as $v_1 \leq v_2$, the right hand side of the equation is ≤ 0. Thus, the two sides are equal only when $v_1 = v_2 = v_3 \equiv v_P$. Hence, any fixed point of the system must be of the form (v_P, v_P, v_P).

2. Existence of unique fixed point

At a fixed point (v_P, v_P, v_P), the equation of the system is

$$V_{CC} = v_P + V_T \frac{R_1(1 + \beta) + R_2}{R_3} W\left(a \exp \frac{v_P}{V_T}\right). \tag{17}$$

Let $R = R_1(1 + \beta) + R_2$ and $R' = R + R_3$ for simplicity. Then, we can solve for v_P to get

$$v_P = V_{CC} - V_T \frac{R}{R'} W\left(\frac{I_S R'}{\beta V_T} \exp \frac{V_{CC}}{V_T}\right). \tag{18}$$

Thus, there exists a fixed point, and as $W(x)$ is single-valued, it is unique. Let us also derive a useful equation for future use. Defining $y = (I_S R'/\beta V_T) \exp(V_{CC}/V_T)$ for ease of notation, we have

$$W\left(a \exp \frac{v_P}{V_T}\right) = \frac{R_3}{R'} W(y). \tag{19}$$

B. Jacobian and its Eigenvalues

In order to determine whether the system has a Hopf bifurcation, we need to determine the eigenvalues of the Jacobian. The Jacobian of the system has two terms

$$A = \left(\frac{\partial}{\partial v_n} \frac{\mathrm{d}v_n}{\mathrm{d}t}\right)\Big|_P \tag{20}$$

$$= -\frac{1}{C}\left(\frac{1}{R_1 + R_2} + \frac{W(y)}{R' + R_3 W(y)}\right), \tag{21}$$

$$B = \left(\frac{\partial}{\partial v_{n-1}} \frac{\mathrm{d}v_n}{\mathrm{d}t}\right)\Big|_P \tag{22}$$

$$= -\frac{\beta}{C} \frac{R_1}{R_1 + R_2} \frac{W(y)}{R' + R_3 W(y)}. \tag{23}$$

In terms of these parameters, the Jacobian is

$$J = \begin{bmatrix} A & 0 & B \\ B & A & 0 \\ 0 & B & A \end{bmatrix}. \tag{24}$$

This matrix has the eigenvalues

$$\lambda_1 = A - \frac{B}{2} + \frac{\sqrt{3}}{2} Bi, \tag{25}$$

$$\lambda_2 = A - \frac{B}{2} - \frac{\sqrt{3}}{2} Bi, \tag{26}$$

$$\lambda_3 = A + B. \tag{27}$$

As A and B are real, we have a pair of complex eigenvalues.

C. Existence of Hopf bifurcation

The mathematics behind Hopf bifurcations is well described in textbooks.[1,16] Interested readers can refer them to learn about the existing result of the Hopf bifurcation and its proof. In terms of the system we are analyzing, the existing result translates to the following. The circuit displays a Hopf bifurcation if it has a fixed point of the node voltage (v_P, v_P, v_P), and at this fixed point, there

exists a critical value R_c of the bifurcation parameter R_3 at which the eigenvalues of the Jacobian satisfy the following properties:

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0 \quad \text{and} \quad \mathrm{Re}(\lambda_3) \neq 0, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}R_3}(\mathrm{Re}(\lambda_1)) = \frac{\mathrm{d}}{\mathrm{d}R_3}(\mathrm{Re}(\lambda_2)) \neq 0. \tag{29}$$

As we saw in Subsection IV A, the system does indeed have a fixed point. Also, as A, B are negative, $\lambda_3 = A + B \neq 0$, which satisfies part of the first condition. For the rest, it is evident that the existence of Hopf bifurcation depends on

$$A - \frac{B}{2} = -\frac{1}{C(R_1 + R_2)}\left[1 + \left(R_1 + R_2 - \frac{\beta R_1}{2}\right) \times \frac{W(y)}{R' + R_3 W(y)}\right]. \tag{30}$$

Let us check the second condition first. Computing the derivative, we get

$$\frac{\mathrm{d}}{\mathrm{d}R_3}\left(A - \frac{B}{2}\right) = \frac{R_1 + R_2 - \frac{\beta R_1}{2}}{C(R_1 + R_2)} \frac{W(y)^2}{(R' + R_3 W(y))^2} \times \frac{2 + W(y)}{1 + W(y)}. \tag{31}$$

As $W(x) > 0 for x > 0$, this derivative is non-zero for all values of R_3—that is, the second condition holds for any R_3. For the first condition, let us define $R_0 = \beta R_1/2 - R_1 - R_2 - R_3$. Then, $A = B/2$ implies

$$W(y) = \frac{R'}{R_0} = \frac{3\beta R_1}{2R_0} - 1. \tag{32}$$

After some algebraic manipulations, this becomes

$$R_3 = \frac{\beta R_1}{2}\left(1 - \frac{3}{W\left(\frac{3I_S R_1}{2V_T}\exp\frac{V_{CC} + V_T}{V_T}\right)}\right) - R_1 - R_2. \tag{33}$$

Thus, we get a critical value of R_3, and hence, the system displays a Hopf bifurcation. For future use, the above equation can be inverted to get V_{CC}

$$\frac{V_{CC}}{V_T} = \frac{3}{1 - \frac{2}{\beta}\frac{R_3 + R_1 + R_2}{R_1}} + \log \frac{\frac{2V_T}{I_S R_1}}{1 - \frac{2}{\beta}\frac{R_3 + R_1 + R_2}{R_1}} - 1. \tag{34}$$

From this, as $W(x) > 0 \; for \; x > 0$, we also get the inequality

$$\beta > 2\frac{R_3 + R_1 + R_2}{R_1}. \tag{35}$$

Thus, this inequality holds if and only if the system displays a Hopf bifurcation at that value of R_3 (Fig. 3).

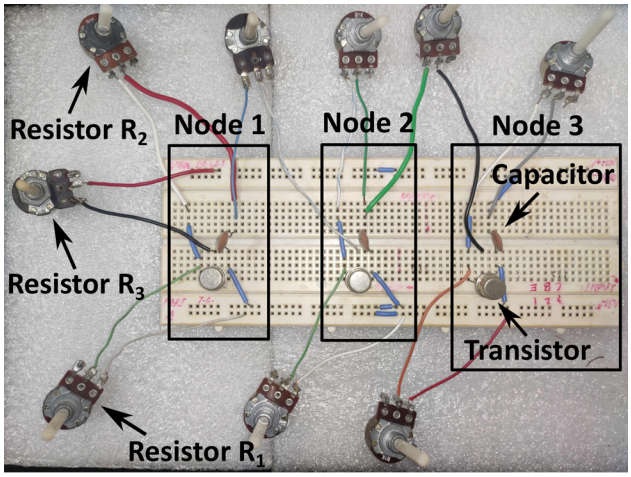

Fig. 3. Photograph of the actual circuit prepared on a breadboard showing all three nodes with electronic components.

V. EXPERIMENTAL SETUP

We constructed the circuit as shown in Fig. 2 with the following circuit components:

- R_1: 1kΩ potentiometer
- R_2: 1kΩ potentiometer
- R_3: 100kΩ potentiometer
- C: 220 nF capacitor
- Transistor (NPN) model: SL100

Note that even though the resistance values R_1 and R_2 are held constant during the experiment, we have used potentiometers for them. This is to ensure that the values of R_1 and R_2 are exactly 1kΩ and, thus, reduce the error in the voltages measured. We used an ordinary lab digital storage oscilloscope (DSO) to measure the voltages v_i of the capacitor as a function of the different values of resistance R_3 and applied voltage V_{CC}. As it was easier to change V_{CC} as compared to R_3, we measured v_i on varying V_{CC} at a fixed R_3, for multiple values of R_3.

VI. RESULTS AND DISCUSSIONS

A. Preliminary results

1. *Verification of active mode of transistors*

In each measurement, for each transistor, the voltage v_{BE} was around $0.65 - 0.75$ V, while the voltage v_{CB} was positive. So, the transistor is indeed operating in the active mode.

2. β value of transistors

The β value of the transistor of each node is given in Table I. These values are not equal, possibly due to manufacturing defects. Thus, we use the mean value of $\beta = 130.5$.

Table I. β values of transistors of the circuit.

Node 1	Node 2	Node 3	Mean	Standard deviation	% Error
129.6	131.7	130.2	130.5	1.08	0.83

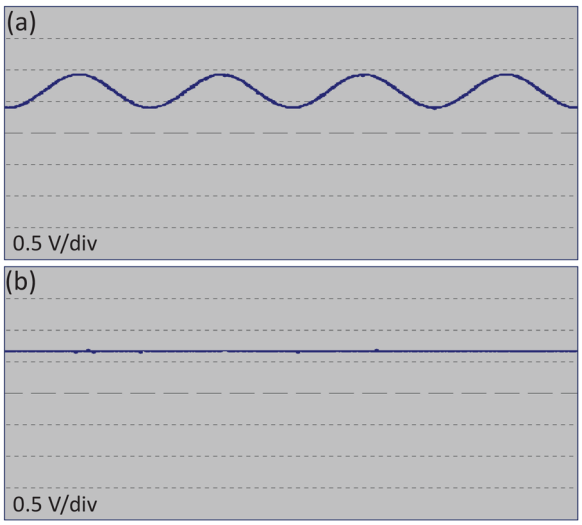

Fig. 4. Hopf bifurcation at node 2 for a bifurcation resistance $R_3 = 61.0k\Omega$ by varying V_{CC}. Waveforms in a digital oscilloscope at (a) voltage before the Hopf bifurcation and (b) afterwards.

B. Existence of Hopf bifurcation and variation with applied voltage

On fixing a value of R_3, we varied V_{CC} applied to the circuit, in order to find its value at which we have a Hopf bifurcation. Figure 4 shows the Hopf bifurcation on a digital oscilloscope at node 2 of the circuit by varying V_{CC} with $R_3 = 61.0k\Omega$. The experimental values of R_3 and V_{CC} are listed in the Table II. From the data, we can see that there is no Hopf bifurcation at $R_3 = 64k\Omega$, as we have

$$2\frac{R_3 + R_1 + R_2}{R_1} = 2\frac{64 \times 10^3 + 10^3 + 10^3}{10^3} = 132 > \beta = 130.5, \tag{36}$$

which satisfies the condition of inequality (Eq. (35)). For smaller values of R_3, Hopf bifurcation occurs. In Fig. 5, we have plotted these values and compared them to the theoretical curve. In Fig. 5, we see that the experimental data closely matches with the theoretical curve constructed using Eq. (34). This supports the theoretical result we had derived previously.

C. Type of Hopf bifurcation

There are two types of Hopf bifurcations: supercritical and subcritical. The most important difference between them, for us, is reversibility. That is, suppose we vary the bifurcation resistance R_3 such that the bifurcation occurs and then go back to the original value of R_3. Then, the system displaying supercritical Hopf bifurcation will revert back to the original voltage state, while it will not for a subcritical Hopf bifurcation. On performing this test to the circuit, we observed that for each value of applied voltage, the fixed-point node voltage returned to the original value. Thus, the circuit displays a supercritical Hopf bifurcation.

Table II. Bifurcation resistance vs applied voltage variation.

Resistance at bifurcation (R_3) ($k\Omega$)	Applied voltage (V_{CC}) (V)
64.0	No bifurcation occurs
62.5	7.4
62.0	4.2
61.5	3.4
61.0	2.9
60.5	2.4

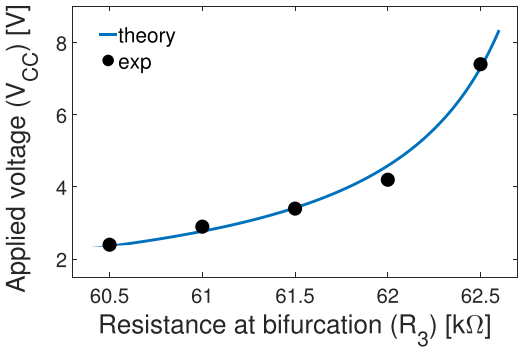

Fig. 5. Applied voltage (V_{CC}) vs resistance at bifurcation (R_3). R^2 value of this plot is 0.9887.

D. Variation of fixed point voltage with applied voltage and resistance

In the fixed point regime of the circuit, we varied the resistance R_3 and applied voltage V_{CC} and observed the values of v_P obtained at each node. From the readings, we can see that the voltages in each of the three nodes are not identical. This can be attributed to the three nodes being dissimilar, which occurs due to the transistors having different β values, for reasons mentioned before. We also plotted the variation of v_P with V_{CC} at a fixed resistance R_3 and compared it with the theoretical curve (obtained from Eq. (18)) in Fig. 6. We can see that there is an excellent match between experimental data and theoretical results, thus supporting the results we have obtained.

VII. CONCLUSION

In this paper, we present a simple electronic circuit consisting of three nodes and a feedback loop, which is similar to a genetic repressilator. The circuit undergoes a Hopf

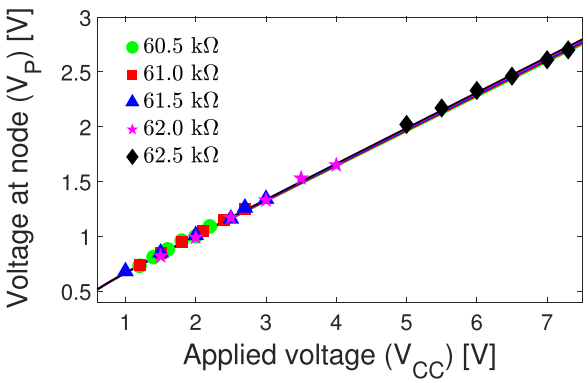

Fig. 6. Plots of fixed point voltage (V_P) with applied voltage (V_{CC}) for different bifurcation resistances (R_3). The solid curve represents the theoretical result given in Eq. (18), while the data points represent the experimentally obtained values. The R^2 values of all five graphs range from 0.9753 to 0.9935.

bifurcation based on various system parameters. DC analysis of the circuit tells us how the voltage (amplitude) varies with the resistance (bifurcation parameter) in the limit cycle close to the bifurcation point. Since all components of the apparatus involved are readily available, and prerequisite knowledge is within the level of a third year college education in physics, this experiment serves as a good introduction for students to experimental nonlinear dynamics.

ACKNOWLEDGMENTS

This study was supported by the Modern Physics Laboratory of the Department of Physics, Indian Institute of Technology Kanpur and Science and Engineering Research Board (SERB) New Delhi (Project No. CRG/2019/000915). The authors thank Upendra Kumar Parashar for valuable help with the experiments.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts of interest to disclose.

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Thunder Houses

These three tiny thunder houses were in the demonstration apparatus collection of the University of Maine when I visited in 2011. In these examples, the spark from a Leiden jar travels down a wire to a spark gap that is near a small amount of explosive material (often dampened gunpowder) in the middle of the house. When this explodes, the top blows off, and we see the reason for installing lightning rods on a house. (Picture and text by Thomas B. Greenslade, Jr., Kenyon College)